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UNCERTAINTY AND OPTIMAL POLICY INTENSITY IN FISCAL AND INCOMES POLICIES

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Two simple linear difference equation models are used to illustrate the effect of uncertainty on optimal fiscal and incomes policy behavior. Whether uncertainty induces more, less, or equally vigorous responses than those of the corresponding deterministic models is shown to depend on both the location of the uncertainty in the underlying linear models and the structure of the criterion functions.

In particular the more complex criterion function characteristic of the incomes policy model gives rise to conclusions regarding relative policy intensity which are quite dissimilar to those obtained for fiscal policy. The conditions under which these conclusions hold for both the fiscal and incomes policy models are derived using control theory. Finally, some simulation results are presented to provide a feel for the quantitative importance of the study's findings.

I. INTRODUCTION

A great deal of study has recently been devoted to the relationship between uncertainty and optimal macroeconomic stabilization policies. At least four questions have been identified and explored. (1) Given both additive and multiplicative uncertainty, is the system (macro model) inherently stabilizable? Assuming a satisfactory answer to this existence query, three other questions can be posed. (2) Does the existence of uncertainty influence the suitable choice of policy instruments? (3) Does the optimal policy derived for a stochastic model yield a significant welfare gain over the policy appropriate to the corresponding deterministic model? and (4) How does uncertainty affect the intensity or vigor with which a particular policy should be employed?

While these last three questions are interrelated and have, in fact, been jointly discussed (see e.g. Turnovsky [8]) this paper focuses more or less exclusively on the final question, with the intent of clarifying and extending the studies of Aoki [1], Brainard [3], Chow [4, 5] and Wonham [9].

The two models considered in this study, one a simple fiscal policy model and the other a simple wage-price control model, are of the same general linear-quadratic form given by

$$\text{Min } D = E\left\{\sum_{t=1}^{T} \tfrac{1}{2}q_t\tilde{x}_t^2 + s_t\tilde{x}_t u_t + \tfrac{1}{2}r_t u_t^2\right\}. \tag{1}$$

Subject to

$$\tilde{x}_{t+1} = \tilde{a}_t x_t + \tilde{b}_t u_t + \tilde{c}_t \tag{2}$$

where x_t defines the state of the system at time t and u_t represents the policy variable. The mean and the variance of the stochastic coefficients $\tilde{a}_t$, $\tilde{b}_t$ and $\tilde{c}_t$ are

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assumed to be known and the random coefficients themselves to be temporally independent although possibly jointly distributed. Furthermore the parameters a_t, b_t and r are assumed to be strictly positive.

In the more conventional fiscal policy model, the inner product term of (1) is assumed to vanish, i.e. $s = 0$. In this event the paper's findings can be summarized as (i) if the uncertainty is additive, i.e. restricted to the c_t term, the first period certainty equivalence theorem holds, (ii) if the noise is multiplicative and restricted to the coefficient of the policy variable, i.e. to $\tilde{b}_t$, a *less* vigorous (than the certainty equivalent) policy is indicated, (iii) if the uncertainty is multiplicative and restricted to the coefficient of the state variable, i.e. to $\tilde{a}_t$, a *more* vigorous policy is implied. While these results are not inconsistent with those obtained in the studies cited, Chow and Turnovsky in particular appear to follow Brainard's lead in attributing any *overall* increase in policy response to covariation between $\tilde{a}_t$ and $\tilde{b}_t$ (or $\tilde{b}_t$ and $\tilde{c}_t$ in Brainard's analysis), rather than to conclusion (iii) above. Since this result is independent of any covariation it represents a separate cause.

For optimization studies in which the inner product term plays an essential role, i.e. in which $s \neq 0$, these findings must be modified. If $s < 0$, as in the incomes policy model outlined below, findings (i) and (iii) survive intact. However, in this new situation, the impact of noise restricted to the policy variable coefficient is no longer unambiguous. In fact, for the particular wage-price control model considered it is possible to establish that, for certain plausible values of s, the indicated policy response is *less* vigorous than the certainty equivalent policy (as above), but for certain other almost equally plausible values of s, the indicated policy response is *more* vigorous.

When $s > 0$, the policy implications of uncertainty are even more qualified although again not indeterminate. In this situation uncertainty restricted either to $\tilde{a}_t$ or $\tilde{b}_t$ can induce either more or less vigorous policy responses than the certainty equivalent ones depending on the relative values of the parameters a, b, r and s.

II. The Fiscal Policy Model

To illustrate the impact of uncertainty on optimal policy decisions we first examine a rather standard single equation macro model in which there is no long term growth, and in which consumption demand in any period C_t^D is related to that period's national income Y_t, and investment demand I_t^D is given autonomously. The demand for government expenditures G_t^D is assumed to consist of two components; G^*, the long run equilibrium level of expenditure which is consistent with the desired public–private expenditure mix and is given autonomously, and g_t, the level of planned (demanded) expenditures for stabilization purposes.[1]

[1]We assume implicitly that g_t is the only policy variable and also that G^* can be given as some fraction of the full (high) employment national income Y^*, i.e. $G^* = kY^*$, and that the prevailing tax structure generates receipts equal to G^* at the full employment national income. We also assume that the given consumption and investment functions are consistent with this tax structure, which is assumed invariant and therefore not a policy option.

The demand structure can thus be given by

$$(3) \qquad \begin{aligned} C_t^D &= \alpha_t + \beta_t Y_t \\ I_t^D &= \gamma_t \\ G_t^D &= G^* + g_t. \end{aligned}$$

Supply is assumed to respond to aggregate excess demand and is thus given by

$$(4) \qquad Y_{t+1} = Y_t + \eta[(C_t^D + I_t^D + G_t^D) - Y_t],$$

where $0 < \eta \leq 1$ and denotes the response coefficient. Combining the supply and demand relationships yields the single equation macro model

$$(5) \qquad Y_{t+1} = a_t Y_t + b_t g_t + \eta(\alpha_t + \gamma_t + G^*),$$

where

$$a_t = 1 - \eta + \eta\beta_t \quad \text{and} \quad b_t = \eta.$$

From (5) we conclude that if $g_t = 0$, $a_t < 1$, and the coefficients α_t, β_t and γ_t are time invariant, the long run equilibrium national income Y' is given by

$$Y' = \frac{\alpha_t + \gamma_t + G^*}{1 - \beta_t}.$$

Furthermore we assume that the price mechanism, defined to include interest and wage rate adjustments, operates to insure that long run equilibrium consumer and investor behavior is consistent with a full (high) employment national income. That is, we assume that when the parameters of the system take on their long run equilibrium values (i.e. when $\alpha_t = \alpha^*$, $\beta_t = \beta^*$, and $\gamma_t = \gamma^*$) for a sustained period, the induced Y' is the full employment or targeted national income Y^*, where

$$Y^* = \frac{\alpha^* + \gamma^* + G^*}{1 - \beta^*}.$$

This implies that the targeted national income is not only consistent with the model, but also that the stabilization problem is essentially a disequilibrium one and arises only when 'short run' behavior deviates from long run equilibrium behavior.

With this in mind we rewrite equation (5) in deviation form as

$$(5') \qquad y_{t+1} = a_t y_t + b_t g_t + c_t,$$

where $\quad y_t \equiv Y_t - Y^*,$

$$c_t = \eta(\Delta\alpha_t + \Delta\gamma_t) + \frac{\eta(\Delta\beta_t)(\alpha^* + \gamma^* + G^*)}{1 - \beta^*},$$

and $\quad \Delta\alpha_t = \alpha_t - \alpha^*$, etc. . . .

Finally, to complete our model we assume we wish to minimize social disutility as measured by the quadratic function

$$(6) \qquad D = \sum_{t=1}^{T+1} \tfrac{1}{2} y_t^2 + r \sum_{t=1}^{T} \tfrac{1}{2} g_t^2.$$

If $\Delta\alpha_t = \Delta\beta_t = \Delta\gamma_t = 0$ for the planning horizon and for the immediate preceding periods, then no corrective stabilization policy is necessary. However, if this is not the case and if the trajectories of these variables, or alternatively of a_t, b_t and c_t, are known with certainty, the problem can be solved by a straightforward application of dynamic programming or the Maximum Principle.

We now assume that the trajectories a_t, b_t and c_t are given only stochastically, with known means and variances. An explicit analytic solution is available, using the same techniques, if the random coefficients (variables) are assumed temporally independent. The stochastic problem can be restated as

$$\text{(7)} \qquad \text{Min } D = E\left\{\frac{1}{2}\sum_{t=1}^{T+1} \tilde{y}_t^2 + \frac{1}{2} r \sum_{t=1}^{T} g_t^2\right\}$$

subject to

$$\text{(8)} \qquad y_{t+1} = \tilde{a}_t y_t + \tilde{b}_t g_t + \tilde{c}_t$$

In addition we note that

$$E(\tilde{y}_t)^2 = \bar{y}_t^2 + \sigma_{y_t}^2,$$

and

$$\bar{y}_{t+1} | y_t = \bar{a}_t y_t + \bar{b}_t g_t + \bar{c}_t,$$

and

$$\text{(9)} \qquad \sigma_{y_{t+1}}^2 | y_t = \sigma_{a_t}^2 y_t^2 + \sigma_{b_t}^2 g_t^2 + \sigma_{c_t}^2.$$

In the final equation above we have assumed solely for the sake of exposition that there is no covariation between $\tilde{a}_t$, $\tilde{b}_t$, and $\tilde{c}_t$. This assumption is relaxed in the more comprehensive formulation included in the appendix.

The general solution to the problem defined by (7), (8) and (9) can be derived from the appendix and is of the form

$$\text{(10)} \qquad g_t = -[K_{t+1}(\bar{b}^2 + \sigma_b^2) + r]^{-1}[K_{t+1}\bar{b}(\bar{a}y_t - \bar{c}_t) + \bar{b}k_{t+1}],$$

where K_t and k_t are the solutions to the Ricatti and tracking equations given in (11a) in the Appendix. Assuming that $K_{T+1} = 1$ and $k_{t+1} = 0$, then

$$\text{(11)} \qquad g_T = -\frac{\bar{b}_T(\bar{a}_T y_T + \bar{c}_T)}{\bar{b}_T^2 + \sigma_b^2 + r},$$

and

$$\text{(12)} \qquad g_{T-1} = -\frac{\{(\bar{b}_{T-1}^2 + \sigma_b^2 + r)(1 + \bar{a}_{T-1}^2 + \sigma_a^2) - \bar{a}_{T-1}^2 \bar{b}_{T-1}^2\}\bar{b}_{T-1}(\bar{a}_{T-1}y_{T-1} + \bar{c}_{T-1}) + \bar{a}_T\bar{b}_T\bar{c}_T(\sigma_b^2 + r)}{\{(\bar{b}_{T-1}^2 + \sigma_b^2)(1 + \bar{a}_{T-1}^2 + \sigma_a^2) + r\}(\bar{b}_{T-1}^2 + \sigma_b^2 + r) - \bar{a}_{T-1}^2\bar{b}_{T-1}^2(\bar{b}_{T-1}^2 + \sigma_b^2)}.$$

Having established the structure of the optimal policy rule, we return now to our primary concern which is to study the impact of uncertainty on optimal policy behavior. To do this we examine the policy rule given by (12), bearing in mind that while the general form of this rule is time invariant, the precise specification is not.

We consider first the impact of uncertainty associated with the additive term c_t and measured by the variance $\sigma^2_{c_T}$. We note that $\sigma^2_{c_T}$ does not appear in the policy rule given by (12). We conclude from this that additive uncertainty does not influence the optimal policy decision. Indeed, if we assume that $\sigma^2_a = \sigma^2_b = 0$, then the policy rule of (12) reduces to

$$(13) \qquad g_{T-1} = -\frac{b_{T-1}(b^2_{T-1} + r + a^2_{T-1}r)(a_{T-1}y_{T-1} + \bar{c}_{T-1}) + a_T b_T \bar{c}_T r}{b^2_{T-1}(b^2_{T-1} + 2r + a^2_{T-1}r) + r^2}.$$

This result is identical to the corresponding certainty solution except that $\bar{c}_T$ and $\bar{c}_{T-1}$ replace c_T and c_{T-1} respectively, and as such implies first period certainty equivalence.

We next address the case in which uncertainty is associated with the response of the economy to the stabilization measure, which implies that $\bar{b}_t$ is known only stochastically. We assume all other relationships are known with certainty in which case (12) reduces to

$$(14) \qquad g_{T-1} = -\frac{\{\bar{b}^2_{T-1} + \sigma^2_b + r + a^2_{T-1}(\sigma^2_b + r)\}\bar{b}_{T-1}(a_{T-1}y_{T-1} + c_{T-1}) + a_T\bar{b}_T c_T(\sigma^2_b + r)}{(b^2_{T-1} + \sigma^2_b)\{b^2_{T-1} + \sigma^2_b + 2r + a^2_{T-1}(\sigma^2_b + r)\} + r^2}$$

which can be rewritten as

$$(14') \qquad g_{T-1} = -\nu_1 y_{T-1} - \nu_2 c_{T-1} - \nu_3 c_T.$$

To examine the impact of this form of uncertainty we differentiate the coefficient of the first term of (14′) with respect to the variance of the policy parameter. This yields

$$(15) \qquad \frac{\partial \nu_1}{\partial \sigma^2_b} = -\frac{a\bar{b}\{(\bar{b}^2 + a^2r)(2r + a^2r) + r^2 + \bar{b}^4 + 2(1 + a^2)(\bar{b}^2 + r + a^2r)\sigma^2_b + (1 + a^2)^2(\sigma^2_b)^2\}}{[(\bar{b}^2 + \sigma^2_b)\{\bar{b}^2 + \sigma^2_b + r + a^2r + a^2\sigma^2_b\} + r(\bar{b}^2 + \sigma^2_b + r)]^2} < 0.$$

The implication of this is immediate. *Ceteris paribus*, any increase in the uncertainty of $\bar{b}_t$ reduces the intensity of the corrective action associated with any given GNP gap.

We next consider the situation in which $\sigma^2_b = \sigma^2_c = 0$, i.e. in which only the state variable parameter, $\tilde{a}_t$, is stochastic. Under these circumstances equation (12) reduces to

$$(16) \qquad g_{T-1} = -\frac{\{(b^2_{T-1} + r)(1 + \sigma^2_a) + \bar{a}^2_{T-1}r\}b_{T-1}(\bar{a}y_{T-1} + c_{T-1}) + \bar{a}_T b_T c_T r}{b^2_{T-1}\{(b^2_{T-1} + r)(1 + \sigma^2_a) + \bar{a}^2 r\} + r(b^2_{T-1} + r)}$$

which can also be rewritten as

$$(16') \qquad g_{T-1} = -\mu_1 y_{T-1} - \mu_2 c_{T-1} - \mu_3 c_T.$$

To measure the impact of this type of uncertainty, we differentiate the coefficient of the first term of (16′) with respect to σ^2_a and obtain

$$(17) \qquad \frac{\partial \mu_1}{\partial \sigma^2_a} = \frac{\bar{a}br(b^2 + r)^2}{[b^2\{(b^2 + r)(1 + \sigma^2_a) + \bar{a}^2 r\} + r(b^2 + r)]^2} > 0.$$

Again the implication is clear. *Ceteris paribus*, uncertainty associated with the parameter $\tilde{a}_t$ increases the vigor or intensity of optimal corrective measures associated with any specific GNP gap. This somewhat unexpected conclusion appears to warrant further examination.

If $r>0$, the optimal *deterministic* policy of period t does not close entirely the GNP gap in period $t+1$. While a further reduction in the GNP gap would be beneficial, the marginal costs in terms of the more intensive policy required to reduce the gap further would more than offset the marginal potential benefits. When a stochastic element is introduced into the $\tilde{a}y_t$ term, a further benefit accrues however which upsets this *deterministic* equilibrium. This additional benefit derives from a reduction in the uncertainty *in period* $t+2$, i.e. in $\sigma^2_{y_{t+2}}$ which as we have noted in (9) above is a concomitant of any reduction in the absolute value of y_{t+1}. Consequently, *ceteris paribus* a more intensive stabilization policy is required to achieve the stochastic equilibrium. We note that this result depends crucially on the dynamic nature of the model and the eminently reasonable assumption that $r>0$.

In summary then we have demonstrated that uncertainty associated with the parameters $\tilde{a}_t$, $\tilde{b}_t$, and $\tilde{c}_t$ in our fiscal policy model tends to induce optimal policy responses which are respectively of greater, lesser and equal intensity than the corresponding optimal deterministic policies. This conclusion requires that $\tilde{a}_t$ and $\tilde{b}_t$ are both positive.

III. The Incomes Policy Model

In this section policy rules for a temporary incomes policy are identified and examined. These rules are optimal with respect to the posited criterion function and a simple but plausible wage-price inflation model. The principal assumption of the underlying model is that the relevant inflation is *sustained* by an inflationary psychology, characterized by expectations of continuing price and wage increases. Temporary wage-price controls by dampening these expectations serve to reduce the inflation.

The inflation model considered is a rather conventional two equation system, consisting of a wage formation equation and a price formation equation. Money wage increases W_t are assumed to be related to expected price increases P^e_t, average productivity increases W'_t, and lagged excess demand for labor as measured by the difference between the prevailing unemployment rate U_{t-1} and the targeted unemployment rate U^*_{t-1}. This relationship is given in equation (18) below with all variables expressed in percentage terms as

$$W_t = P^e_t + W'_t + \eta(U_{t-1} - U^*_{t-1}). \tag{18}$$

If we make the additional assumption that expected price increases are related to past price changes P_t as given by

$$P^e_t = h(P_{t-1} + dP_{t-2} + d^2P_{t-3+\ldots}), \tag{19}$$

where $0<h<1$ and $d=1-h$, we can use a Koyck transformation to obtain the following *wage (increase) formation equation*

$$W_{t+1} = dW_t + hP_t + W'_{t+1} - dW'_t + \eta(U_t - U^*_t) - \eta d(U_{t-1} - U^*_{t-1}). \tag{20}$$

On the other hand, *price increases* are assumed to be related to increases in average unit labor costs, $\bar{\bar{W}}_t = W_t - W'_t$, and to aggregate excess demand, $Y_t - Y^*_t$, where Y^*_t equals the output defined by operating at the targeted unemployment rate U^*_t and the targeted capital capacity level. A closed economy is assumed in the sense that no allowance is made for disproportionate exogenous price increases. If we assume, in addition, that the relationship between price changes and past wage changes follows a Koyck lag structure, it follows that

$$P_t = b(\bar{\bar{W}}_{t-1} + a\bar{\bar{W}}_{t-2} + a^2\bar{\bar{W}}_{t-3} + \ldots) + \omega(Y_t - Y^*_t), \tag{21}$$

where $0 < b < 1$ and $a = 1 - b$. This implies the *price formation equation,*

$$P_{t+1} = aP_t + bW_t - \omega(Y_t - Y^*_t) - \omega a(Y_{t-1} - Y^*_{t-1}). \tag{22}$$

It is evident from (20) and (22) that the inflationary process defined in this study is self-feeding, i.e. is characterized by a wage-price spiral, and furthermore that this process can be interrupted only by (i) a substantial increase in either or both the unemployment gap $U_t - U^*_t$ and the deflationary gap $Y^*_t - Y_t$, or by (ii) wage-price controls which negate either the wage formation equation (20) or the price formation equation (22). We note also that the system given by (20) and (22) is capable of generating an unstable wage-price spiral, i.e., one characterized by a continuing escalation in the rate of increase (decrease) of prices and wages. However, if the system is constrained so that $U_t = U^*_t$ and $Y_t = Y^*_t$, $t = (1, 2 \ldots)$, the wage-price spiral is not only stable, but the equilibrium values of P_t and W_t depend only on the initial values of those same variables.

In the analysis which follows we assume that the system is so constrained, i.e. that the economy is operating at the target levels of output and employment either with or without the assistance of monetary and fiscal policies. Consequently, the final two terms on the r.h.s. of equations (20) and (22) can be eliminated, and the inflation is thus sustained only by inflationary expectations. In this situation wage and/or price controls are designed to combat the inflation by altering these expectations. We assume further that direct wage controls are imposed; in which case the inflation model reduces to

$$P_{t+1} = aP_t + b\bar{\bar{W}}_t \tag{23}$$

where $\bar{\bar{W}}_t \equiv W_t - W'_t$ and represents the policy or control variable, while P_t is determined in the market.

A good incomes policy must provide for (i) the elimination of price inflation or at least its reduction to some acceptable level, (ii) an equitable distribution of any restraining impact on wages, interest rates, and profits, (iii) terminal characteristics which minimize the possibility of the reintroduction of a continually escalating wage-price spiral once controls are suspended. This last objective requires that the terminal price expectation P^e_{T+1} is equal to the targeted level of price increase P^*, which may or may not be equal to zero. A criterion function constructed to achieve these three objectives when it is minimized is given by

$$D = \sum_{t=1}^{T} \{\rho_1|P_t - P^*| + |W_t - (W' + P_t)|\} + \rho_2|P^e_{T+1} - P^*|. \tag{24}$$

A formal statement of the control problem requires one additional observation. The terminal condition as given by the third term of (24) is expressed in terms of the expected price increase P^e_{T+1} which is itself determined by a price increase trajectory as per equation (19). This relationship permits us to combine the first and third terms of (24) and to rewrite the criterion function as

$$(25) \qquad D=\sum_{t=1}^{T} \rho_t|P_t-P^*|+|(W_t-W_t')-P_t|$$

where $\rho_t \equiv \rho_1+hd^{(T-t)}\rho_2$ for $t=(1,2,\ldots,T)$.

Finally we find it convenient to replace the absolute value form of the criterion function (25) with the mathematically more tractable quadratic structure of (26). If we simultaneously set $P^*=0$ and normalize on the state vector coefficient we can rewrite (25) as

$$(26) \qquad D=\sum_{t=1}^{T} \tfrac{1}{2}P_t^2-s_t\bar{\bar{W}}_tP_t+\tfrac{1}{2}r_t\bar{\bar{W}}_t^2,$$

where $s_t=r_t$.

Minimizing (26) subject to (23) yields a *variable coefficient policy* rule of the form

$$(27) \qquad W_t=\theta_tP_t+W_t',$$

where $0\le\theta_t\le 1$ and where $\theta\to 1$ as $t\to T$. This provides for a reasonable 'reentry' into the market.

The stochastic version of the same model can be restated as

$$(28) \qquad \text{Min } D=E\left\{\sum_{t=1}^{T} \tfrac{1}{2}\tilde{P}_t^2-s_t\tilde{P}_t\bar{\bar{W}}_t+\tfrac{1}{2}r_t\bar{\bar{W}}_t^2\right\}$$

subject to

$$(29) \qquad \tilde{P}_{t+1}=\tilde{a}_tP_t+\tilde{b}_t\bar{\bar{W}}_t+\xi_t,$$

where ξ_t is assumed to be temporally independent and to have a zero mean.

From the appendix we see that if we exclude any covariation, the general form of the solution to the problem given by (28) and (29) is

$$(30) \qquad \bar{\bar{W}}_t=-\frac{\bar{a}\bar{b}K_{t+1}-s_t}{(\bar{b}+\sigma_b^2)K_{t+1}+r_t}P_t.$$

Assuming for expositional purposes that s and r are time invariant, and also that $K_{T+1}=1$, it follows from (30) that for the final two periods,

$$(31) \qquad \bar{\bar{W}}_T=-\frac{\bar{a}\bar{b}-s}{(\bar{b}^2+\sigma_b^2)+r}P_T,$$

and

$$(32) \qquad \bar{\bar{W}}_{T-1}=-\frac{\bar{a}\bar{b}\{(1+\bar{a}^2+\sigma_a^2)(\bar{b}^2+\sigma_b^2+r)-(\bar{a}\bar{b}-s)^2\}-s(\bar{b}^2+\sigma_b^2+r)}{(\bar{b}^2+\sigma_b^2)\{(1+\bar{a}^2+\sigma_b^2+r)-(\bar{a}\bar{b}-s)^2\}+r(\bar{b}^2+\sigma_b^2+r)}P_{T-1}.$$

It is evident from (31) and (32) that additive uncertainty does not affect the optimal policy rule, i.e. that the certainty equivalent theorem holds in this case.

When the uncertainty is restricted to the coefficient of the state variable, i.e. when $\sigma_b^2 = \sigma_c^2 = 0$, the policy rule given by (32) can be rewritten as

$$(33) \qquad \bar{\bar{W}}_{T-1} = -\kappa P_{T-1} = -\frac{\{\bar{a}\bar{b}F - s(b^2+r)\} + \bar{a}b(b^2+r)\sigma_a^2}{\{b^2F + r(b^2+r)\} + b^2(b^2+r)\sigma_a^2} P_{T-1}$$

where $F = (b^2+r)(1+\bar{a}^2) - (\bar{a}b - s)^2$.

As above, to determine the impact of this uncertainty we differentiate κ with respect to the variance of $\tilde{a}_t$, and obtain

$$(34) \qquad \frac{\partial\kappa}{\partial\sigma_a^2} = \frac{(\bar{a}br + b^2 s)(b^2+r)^2}{[b^2\{(1+\bar{a}^2+\sigma_a^2)(b^2+r) - (\bar{a}b - s)^2\} + r(b^2+r)]^2} > 0.$$

As in section II above this implies a *more* vigorous policy response for uncertainty of this type. However, if the inner product term is *added* as in (1) rather than *subtracted* as in (28), the denominator of (34) is given by $(\bar{a}br - b^2 s)(b^2+r)^2$ and therefore $\partial\kappa/\partial\sigma_a^2 > 0$ only when $s < ar/b$, and negative otherwise. In those instances when $s = r$, the sign of $\partial\kappa/\partial\sigma_a^2$ depends exclusively on whether or not $a > b$.[2]

Returning again to the incomes policy model, it follows from (32) that if uncertainty is restricted to the policy variable, i.e. if $\sigma_a^2 = \sigma_c^2 = 0$, that

$$(35) \qquad \bar{\bar{W}}_{T-1} = -\lambda P_{T-1} = -\frac{\{a\bar{b}F - s(\bar{b}^2+r)\} + \{a\bar{b}(1+a^2) - s\}\sigma_b^2}{\{\bar{b}^2F + r(\bar{b}^2+r)\} + \{F + \bar{b}^2(1+a^2) + r\}\sigma_b^2 + (1+a^2)(\sigma_b^2)^2} P_{T-1}$$

The complexity of (35) precludes finding simple necessary and sufficient conditions for determining the sign of $\partial\lambda/\partial\sigma_b^2$. However, *sufficient* conditions can be readily ascertained. In particular

$$(36) \qquad \partial\lambda/\partial\sigma_b^2 > 0, \text{ when } a\bar{b}(1+a^2) < s < a\bar{b} + [(1+a^2)(\bar{b}^2+r)]^{1/2}.$$

This result is particularly interesting because when (36) holds, the conclusion contradicts the findings of section II. Also since we have already shown that $\partial\lambda/\partial\sigma_a^2$ is positive for the incomes policy model, it follows that uncertainty in either a_t or b_t induces a more vigorous policy response. (Again no covariation is assumed.) Furthermore, it is easy to demonstrate that plausible values of s do satisfy (36). In particular, in the model employed in which $a = 0.77$ and $b = 0.23$ and in which $r = s$, (36) is satisfied whenever $0.23 \leq r \leq 1.98$. In addition we note that this is a sufficient condition and that a larger interval might also satisfy (36).

In the more familiar case in which the inner product term is positive as in (1) a simple sufficiency condition can be derived only for $\partial\lambda/\partial\sigma_b^2 < 0$. For this case

$$(37) \qquad \partial\lambda/\partial\sigma_b^2 < 0, \text{ when } ar/b < s < [(1+a^2)(b^2+r)]^{1/2} - ab.$$

In the common special case when a, $b < 1$, the upper limit implies that s cannot be significantly greater than r, if we wish to guarantee that $\partial\lambda/\partial\sigma_b^2 < 0$. Also for $r = s$,

[2]This result is consistent with our earlier explanation. Increasing a_t while holding b_t constant implies via (1) that x_{t+1} increases, which in turn implies a reduction in the absolute value of $\sigma^2_{x_{t+2}}$.

the lower bound condition of (37) again emphasizes the importance of the relative magnitudes of a and b.

III. Summary

The findings of the previous two sections were obtained for a single equation model with a two period horizon. To illustrate that these conclusions are in general transferable to systems characterized by an n period horizon ($n > 2$) and simultaneous equations, some control simulation were made for a simple 2 equation model with a twleve period horizon. The results are reported in Table 1. The model is defined by

$$A = \begin{pmatrix} 1.25 & -0.75 \\ 1.0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0.5 \\ 0 & 0 \end{pmatrix}, \quad C_t = \begin{pmatrix} -25 \\ 0 \end{pmatrix}, \quad X_0 = \begin{pmatrix} -50 \\ 50 \end{pmatrix}$$

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

All of the results in Table 1 are consistent with the findings in Section II. A comparison of the results of simulations (2) and (3) with those of simulation (1) show that when uncertainty is restricted to the A matrix, that policy is pursued more intensively than for the corresponding certainty case. Similarly a comparison of row 5 with row 1 illustrates the less vigorous response which arises when uncertainty is confined to the B matrix. The results of simulation 4 demonstrate the substitution of policy 1 when uncertainty is restricted to the coefficient of the latter. The other results are self explanatory.

TABLE 1

Uncertainty and First Period Policy Responses

Simulation	Positive Variances	Insentity of Policy Response	
		u_1	u_2
1	none	31.8	15.9
2	$\sigma^2_{a11}, \sigma^2_{a12}$	34.3	17.6
3	$\sigma^2_{a11}, \sigma^2_{a12}, \sigma^2_{a11a12}$	35.1	17.5
4	σ^2_{b11}	23.1	25.0
5	$\sigma^2_{b11}, \sigma^2_{b12}$	26.9	13.5
6	σ^2_{c11}	31.8	15.9
7	*all*	29.5	14.8

When positive the variances have the following magnitudes $\sigma^2_{a11} = 0.25$, $\sigma^2_{a12} = 0.09$, $\sigma^2_{a11a12} = 0.02$, $\sigma^2_{b11} = 0.2$, $\sigma^2_{b12} = 0.2$, $\sigma^2_{c11} = 9.0$.

In summary the primary factors which appear to influence the intensity of policy response in a stochastic model are whether (1) the uncertainty resides primarily in $a(A)$ or $b(B)$, (2) the inner product matrix $s(S)$ is positive, negative or zero and (3) $a > b$ (or in some sense $A > B$). Given this host of qualifications, it would appear prudent to simulate any specific model or system before making any qualitative judgment about uncertainty and policy intensity. Furthermore since policy intensity, welfare gains and policy instrument choice are all interrelated, it seems reasonable to assume that this same course of action might also be advisable prior to any judgment in these latter two areas.

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APPENDIX

DERIVATION OF THE OPTIMAL POLICY RULES FOR A STOCHASTIC LINEAR-QUADRATIC DYNAMIC PROGRAM

The Problem

(1)
$$\text{Minimize } E\left\{\frac{1}{2}\sum_{t=1}^{T} x_t'Q_tx_t + x_t'S_tu_t + \frac{1}{2}\sum_{t=1}^{T} u_t'R_tu_t + \frac{1}{2}x'_{T+1}Q_{T+1}x_{T+1}\right\}$$

(2)
$$\text{s.t. } x_{t+1} = \tilde{A}x_t + \tilde{B}u_t + \tilde{c}_t.$$

A Necessary Relationship

(3)
$$E[x'_{t+1}Qx_{t+1}] = \bar{x}'_{t+1}Q\bar{x}_{t+1} + tr(QV^{x_{t+1}}),$$

where the ijth element of the matrix $V^{x_{t+1}}$ is given by

(4) $$V_{ij}^{x_{t+1}} = x_t'V^{A_iA_j}x_t + 2x_t'V^{A_iB_j}u_t + 2x_t'V^{A_iC_j} + 2u_t'V^{B_iC_j} + u_t'V^{B_iB_j}u_t + V^{C_iC_j}$$

where $V^{A_iB_j}$ = the covariance matrix of the ith row of A with the jth row of B.

Outline of the Dynamic Programming Derivation

Let

(5)
$$F_{T+1}(x_{T+1}) = \tfrac{1}{2}x'_{T+1}Q_{T+1}x_{T+1},$$

and

(6)
$$F_T(x_T) = \min_{u_T}\}E(\tfrac{1}{2}x'_TQ_Tx_T + x'_TSu_T + \tfrac{1}{2}u'_TRu_T + F_{T+1}(x_{T+1}))\}$$

$$= \min_{u_T}\{\tfrac{1}{2}x'_TQ_Tx_T + x'_TSu_T + \tfrac{1}{2}u'_TRu_T + E(\tfrac{1}{2}(x'_{T+1}Q_{T+1}x_{T+1})|x_T)\},$$

which by (2), (3), and (4) yields

$$(6')\quad F_T(x_T)=\operatorname*{Min}_{u_T}\{\tfrac{1}{2}x'_TQ_Tx_T+\tfrac{1}{2}u'_TRu_T+x'_TSu_T$$
$$+\tfrac{1}{2}(\bar{A}x_T+\bar{B}u_T+\bar{c}_T)'Q_{T+1}(\bar{A}x_T+\bar{B}u_T+\bar{c}_T)$$
$$+\tfrac{1}{2}tr(Q_{T+1}(x'_TV^{AA}x_T+2x'_TV^{AB}u_T+2x'_TV^{AC}+2u'_TV^{BC}$$
$$+u'_TV^{BB}u_T+V^{CC}))\}.$$

The value of u_T which minimizes the right hand side of (6′) can be found by setting $\partial\{\ \}/\partial u_T=0$. This yields the optimal policy rule,

$$(7)\quad u_T=-[R_T+\bar{B}'Q_{T+1}\bar{B}+\overline{trQ_{T+1}V^{BB}}]^{-1}\{\bar{B}'Q_{T+1}(\bar{A}x_T+\bar{c}_T)+(\overline{trQ_{T+1}V^{AB}})'x_T$$
$$+S'_Tx_T+\overline{trQ_{T+1}V^{BC}}\}^1.$$

Substituting (7) back into (6′) yields the quadratic form

$$(8)\qquad F_T(x_T)=\tfrac{1}{2}x'_TK_Tx_T+k_Tx_T+h_T,$$

where

$$(8a)\quad K_T=Q_T+\bar{A}'Q_{T+1}\bar{A}+\overline{trQ_{T+1}V^{AA}}-(\bar{A}'Q_{T+1}\bar{B}+S_T+\overline{trQ_{T+1}V^{AB}})[\ ]^{-1}$$
$$\times(\bar{B}'Q_{T+1}\bar{A}+S'_T+(\overline{trQ_{T+1}V^{AB}})')$$
$$k_T=\bar{c}'_TQ_{T+1}\bar{A}+(\overline{trQ_{T+1}V^{AC}})'-(\bar{c}'_TQ_{T+1}\bar{B}+(\overline{trQ_{T+1}V^{BC}})')[\ ]^{-1}$$
$$\times(\bar{B}'Q_{T+1}\bar{A}+S'_T+(\overline{trQ_{T+1}V^{AB}})')$$
$$h_T=\tfrac{1}{2}\{\bar{c}'_TQ_{T+1}\bar{c}_T+\overline{trQ_{T+1}V^{CC}}+(\bar{c}'_TQ_{T+1}\bar{B}+(\overline{trQ_{T+1}V^{BC}})')[\ ]^{-1}$$
$$\times(\bar{B}Q_{T+1}\bar{c}_T+\overline{trQ_{T+1}V^{BC}})\}.$$

The same procedure is repeated for the new recursive equation

$$(9)\quad F_{T-1}(x_{T-1})=\operatorname*{Min}_{u_{T-1}}E\{(\tfrac{1}{2}x'_{T-1}Q_{T-1}x_{T-1}+x'_{T-1}S_{U_{T-1}}+u'_{T-1}Ru_{T-1}+F_T(x_T))\}.$$

The only difference between this new equation and equation (6) above is that $F_T(x_T)$ given by (8) is more complex than $F_{T+1}(x_{T+1})$ given by (5). Consequently, the structure of the optimal policy rule derived from (9) resembles that of equation (7), but is modified to incorporate the difference just cited. The derived optimal policy rule is given as

$$(10)\quad u_{T-1}=-[R_{T-1}+\bar{B}'K_T\bar{B}+\overline{trK_TV^{BB}}]^{-1}\{\bar{B}'K_T(\bar{A}x_{T-1}+\bar{c}_{T-1})$$
$$+S'_{T-1}x_{T-1}+\bar{B}'k_T+(\overline{trK_TV^{AB}})'x_{T-1}+\overline{trK_TV^{BC}}\}$$

The functional equation (9) can then be rewritten as

$$(11)\qquad F_{T-1}(x_{T-1})=\tfrac{1}{2}x'_{T-1}K_{T-1}x_{T-1}+k_{T-1}x_{T-1}+h_{T-1},$$

where

$$(11a)\quad K_{T-1}=[Q_{T-1}+\bar{A}'K_T\bar{A}+\overline{trK_TV^{AA}}$$
$$-W_{T-1}(\bar{B}'K_T\bar{A}+S_{T-1}+(\overline{trK_TV^{AB}})')],$$
$$k_{T-1}=[\bar{A}'-W_{T-1}\bar{B}'](K_T\bar{c}_{T-1}+k_T)+\overline{trK_TV^{AC}}-W_{T-1}\overline{trK_TV^{BC}},$$

with

$$W_{T-1} \equiv (\bar{A}'K_T\bar{B} + S_{T-1} + \overline{trK_TV^{AB}})[R_{T-1} + \bar{B}'K_T\bar{B} + \overline{trK_TV^{BB}}]^{-1}.$$

Since the structural form of (11) is identical to that of (8), a repeated application of the dynamic program analysis yields the same results. Consequently (10) and (11a) constitute the general form of the solution with t and $t-1$ replacing respectively T and $T-1$ everywhere. The values of the shadow price variables, K_t and k_t, for $t=(T, T-1, \ldots, 1)$ can be determined by solving (11a) recursively given the boundary conditions $K_{T+1} = Q_{T+1}$ and $k_{T+1} = 0$. Once these have been identified, the policy rule given by (10) is determined.

[1]If, e.g., both the B and Q matrices are 2×2, then the matrix given by the symbol $\overline{trQ_{T+1}V^{BB}}$ is defined as $\overline{trQV^{BB}} \equiv q_{11}V^{B_1B_1} + q_{12}V^{B_2B_1} + q_{21}V^{B_1B_2} + q_{22}V^{B_2B_2}$.